Lec 13

Linear Transformations

I Given two vector spaces U and V, consider a function

$$T : U \to V$$

between U and V. We shall call T to be a linear transformation if it satisfies ~~the~~ following two properties:

(i) $\forall\ u_1, u_2 \in U$ we have

$$T(u_1 + u_2) = T(u_1) + T(u_2).$$

(ii) $\forall\ u \in U\ \&\ \alpha \in \mathbb{R}$ we have

$$T(\alpha u) = \alpha T(u).$$

One can combine two properties of a linear transformation as follows:

$$\boxed{\begin{gathered}\forall\, u_1, u_2 \in U \ \&\ \alpha_1, \alpha_2 \in \mathbb{R} \text{ we have} \\ T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2)\end{gathered}}$$

Remark:

If $\alpha = 0$ we have

$T(0 \cdot u) = 0 \cdot T(u) \Rightarrow T(0) = 0.$

A linear transformation maps the zero vector to the zero vector.

<u>Example 1</u>:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

We want to show that T is linear:

Let $u_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $u_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$u_1 + u_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$T(u_1) = \begin{pmatrix} x_1 + y_1 \\ x_1 - y_1 \end{pmatrix} \; ; \; T(u_2) = \begin{pmatrix} x_2 + y_2 \\ x_2 - y_2 \end{pmatrix}$$

$$T(u_1) + T(u_2) = \begin{pmatrix} (x_1 + y_1) + (x_2 + y_2) \\ (x_1 - y_1) + (x_2 - y_2) \end{pmatrix}$$

They are equal.

$$T(u_1 + u_2) = \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) \\ (x_1 + x_2) - (y_1 + y_2) \end{pmatrix}$$

$$\alpha u_1 = \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \end{pmatrix} \; ; \; T(\alpha u_1) = \begin{pmatrix} \alpha x_1 + \alpha y_1 \\ \alpha x_1 - \alpha y_1 \end{pmatrix}$$

They are equal.

$$\alpha T(u_1) = \begin{pmatrix} \alpha (x_1 + y_1) \\ \alpha (x_1 - y_1) \end{pmatrix}$$

Hence T is linear.

Example 2 (Reflection about x axis)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

Example 3 (Reflection about y axis)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -x \\ y \end{pmatrix}$$

Example 4 (Reflection about the origin)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Example 5 (Anticlockwise rotation by an angle θ)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by a matrix. For example

Example 6

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} 3 & 4 & 5 & 9 \\ 1 & 2 & 0 & -7 \\ -3 & 0 & 4 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

In the above example T is defined using a 3×4 matrix A. If we write

$$\underline{x} = (x_1 \; x_2 \; x_3 \; x_4)^T$$

then

$$T(\underline{x}) = A\underline{x}$$

Range and Null spaces of T

Let

$$T : U \to V$$

be a linear transformation.

(a) We define "Null Space" of T as follows

$$\mathcal{N}(T) \triangleq \{u \in U : T(u) = 0\}$$

(b) We define "Range Space" of T as follows

$$\mathcal{R}(T) \triangleq \{v \in V : \exists\, u \in U, T(u) = v\}.$$

Fact:

$\mathcal{N}(T)$ is a vector subspace of the domain U.

$\mathcal{R}(T)$ is a vector subspace of the target V.

The linear transformation is called "surjective" or "onto" if

$$\mathcal{R}(T) = V.$$

The linear transformation is called "injective" or "one-one" (1-1) if
$\mathcal{N}(T)$ is trivial ie $\mathcal{N}(T) = \{0\}$.

Fact:

$$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U$$

Example 7

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+2y \\ 2x+4y \end{pmatrix}$$

$$\mathcal{N}(T) \triangleq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x+2y=0,\ 2x+4y=0 \right\}$$

$x = -2y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} y.$$

$$\mathcal{N}(T) \doteq \operatorname{span}\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]$$

Basis of $\mathcal{N}(T) = \left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

$\dim \mathcal{N}(T) = 1$

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Let $v \in \mathcal{R}(T)$, by definition of $\mathcal{R}(T)$ $\exists \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = v$$

writing

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

it follows that

$$T\begin{pmatrix} x \\ y \end{pmatrix} = xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix} = v$$

Hence v can be written as a l.c. of $\longrightarrow$ $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\mathcal{R}(T) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right\}$$

But $\begin{pmatrix}1\\2\end{pmatrix}$ & $\begin{pmatrix}2\\4\end{pmatrix}$ are not linearly independent.

$$R(T) = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$$

Basis of $R(T)$ is $\begin{pmatrix}1\\2\end{pmatrix}$

$$\dim R(T) = 1.$$

T is neither surjective, nor injective.

Also note that

$$T\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix}$$

It is easy to see that the null-space of T is also the null space of the matrix A. This is because if $\begin{pmatrix}x\\y\end{pmatrix} \in \mathcal{N}(T)$, we have

$$T\begin{pmatrix}x\\y\end{pmatrix} = 0 \Rightarrow A\begin{pmatrix}x\\y\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} \text{ is in the null space of } A.$$

$\therefore$ $R(T)$ is spanned by

$$T\begin{pmatrix}1\\0\end{pmatrix} \text{ and } T\begin{pmatrix}0\\1\end{pmatrix}$$

and since

$$T\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix} = 1^{st} \text{ column of } A.$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = A\begin{pmatrix}0\\1\end{pmatrix} = 2^{nd} \text{ column of } A$$

it follows that

$R(T)$ is the column space of the matrix A.

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Theorem:

Every linear transformation is completely defined if it is defined on a set of basis vectors of the domain.

$T: U \to V$

and let

$\{u_1, u_2, \ldots, u_m\}$

be a set of basis vectors of U. Define

$v_j = T(u_j), \; j = 1, 2, \ldots, n.$

The theorem claims that $T(u)$ is defined on any $u \in U$.

Proof: Since $\{u_1, \ldots, u_n\}$ is a basis it follows that given $u \in U$, $\exists$ scalars $\alpha_1, \ldots, \alpha_n$:

$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n.$

Thus

$$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$$
$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n).$$
$$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Example 8:

$T: \mathbb{R}^3 \to \mathbb{R}^5$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\\3\\5\end{pmatrix}; \quad T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}; \quad T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\\1\end{pmatrix}$$

T is defined on a basis

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

of $\mathbb{R}^3$.

We want to calculate

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

The first step is to find scaler α, β, γ:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}1\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Clearly

$$\left.\begin{aligned} \alpha+\beta+\gamma &= x \\ \beta+\gamma &= y \\ \gamma &= z \end{aligned}\right\} \quad \begin{aligned} \gamma &= z \\ \beta &= y-z \\ \alpha &= x-\cancel{z}-y+\cancel{z} = x-y \end{aligned}$$

$\therefore\ \alpha = x-y,\ \beta = y-z,\ \gamma = z.$

It follows that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 5 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x-y \\ 2x-2y \\ y-x \\ 3x-3y \\ 5x-5y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y-z \\ 0 \\ y-z \end{pmatrix} + \begin{pmatrix} 0 \\ z \\ 0 \\ z \\ z \end{pmatrix}$$

Thus

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ 2x-2y+z \\ 2y-x-z \\ 3x-3y+z \\ 5x-4y \end{pmatrix}.$$

To calculate $\mathcal{N}(T)$, we proceed as follows:

$$\mathcal{N}(T) = \{(x,y,z) : x-y=0,\ 2x-2y+z=0,\ 2y-x-z=0,\ 3x-3y+z=0,\ 5x-4y=0\}.$$

$$= \{(0,0,0)\}.$$

$\dim \mathcal{N}(T) = 0.$

T is 1-1 ie injective.

$R(T)$ is spanned by

$$T\begin{pmatrix}1\\0\\0\end{pmatrix},\ T\begin{pmatrix}1\\1\\0\end{pmatrix},\ T\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Hence

$$R(T) = \operatorname{Span}\Bigg\{ \underbrace{\begin{pmatrix}1\\2\\-1\\3\\5\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\1\end{pmatrix}}_{\text{They are independent.}} \Bigg\}.$$

$\dim R(T) = 3$.

T is not onto. The range is a 3 dimensional subspace in $\mathbb{R}^5$.

Example 9

Consider the vector space $P_5(t)$, the space of polynomials of degree ≤ 5. Define

$$T : P_5(t) \longrightarrow P_3(t)$$
$$p(t) \longmapsto \frac{d^2 p(t)}{dt^2}.$$

A polynomial

$$p(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f$$

is mapped to

$$\frac{d^2 p}{dt^2} = 20at^3 + 12bt^2 + 6ct + 2d.$$

via the transformation T. It is easy to verify that T is linear

$$T(p_1 + p_2) = \frac{d^2}{dt^2}(p_1 + p_2) = \frac{d^2}{dt^2}p_1 + \frac{d^2}{dt^2}p_2$$
$$= T(p_1) + T(p_2).$$

Likewise $T(\alpha p) = \alpha T(p)$.

$\mathcal{N}(T)$ is the set of all $p(t)$: $T(p) = 0$

$\Rightarrow 20at^3 + 12bt^2 + 6ct + 2d \equiv 0$

$\Rightarrow a = b = c = d = 0$.

This is the zero polynomial.

Hence $\mathcal{N}(T) = \text{span}\{1, t\}$.

and are of the form $et + f$.

$\dim \mathcal{N}(T) = 2$.

$R(T) = \text{span}\{T(1), T(t), T(t^2), T(t^3), T(t^4), T(t^5)\}$.

$= \text{span}\{0, 0, 2, 6t, 12t^2, 20t^3\}$.

$= \text{span}\{1, t, t^2, t^3\} = P_3(t)$.

$\dim R(T) = 4$.

T is not 1-1, but T is onto.

Example 10

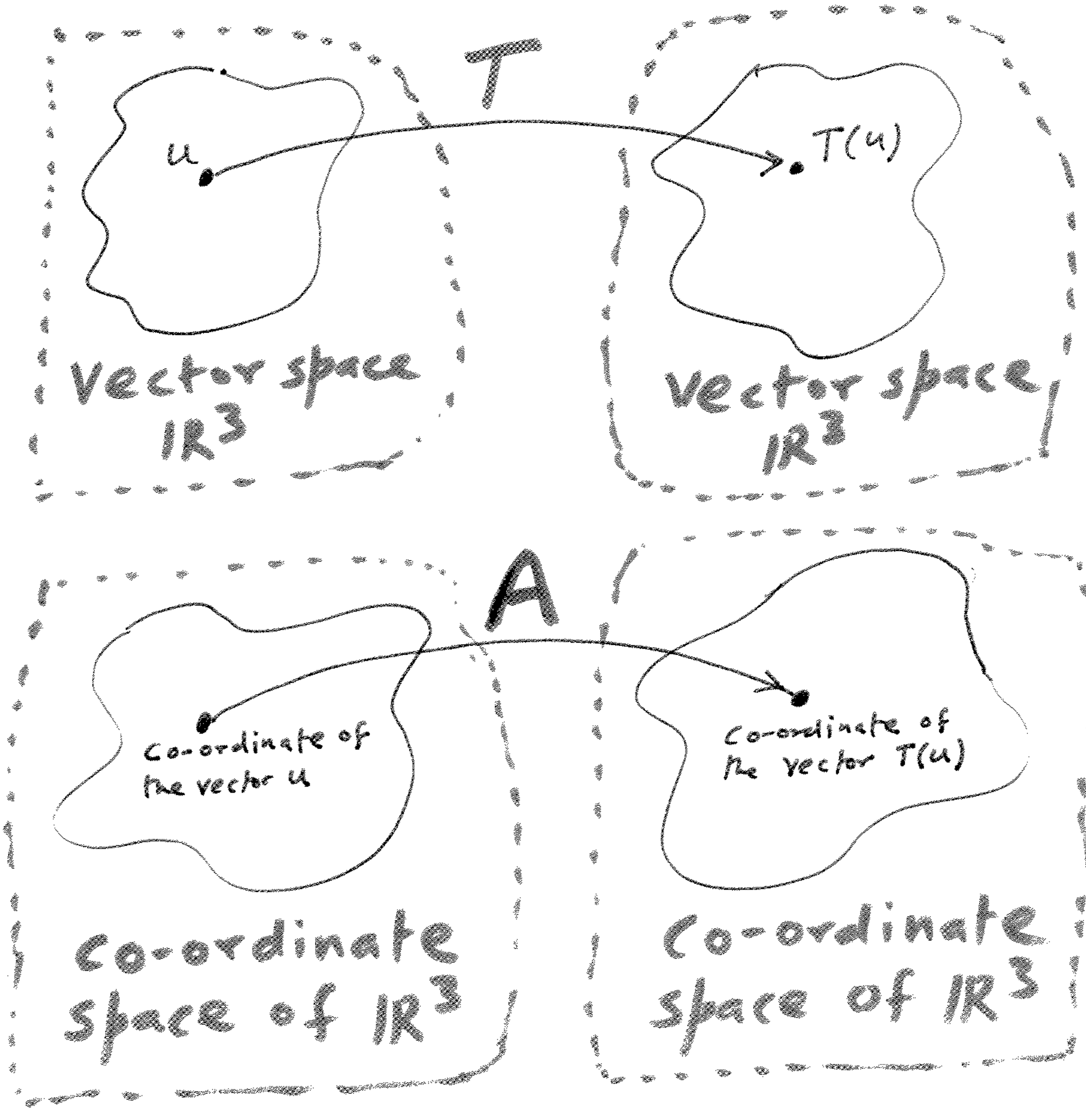
In this example, we consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined as follows:

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ y-z \\ x+2y \end{pmatrix}$$

So far, we had been thinking T as a function that acts on vectors in the domain producing vectors in the target.

For example

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$



Choose a basis B of $\mathbb{R}^3$ as $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\} = B$

In this example, we want to do something different. We want to construct a 3×3 matrix A which has the following property

$$A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \longmapsto A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ are the 3 co-ordinates of 'u' with respect to a fixed basis B.

In this example we have chosen a basis B in the bottom of page 13·19.

Remark: A defines a mapping of co-ordinates and not the vectors of $\mathbb{R}^3$.

" For any vector $u \in \mathbb{R}^3$, matrix "A" operates on the co-ordinate vector $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ of u w.r.t. the chosen basis B and produces the vector $\beta = A\alpha$ where the vector β ~~as~~ has the co-ordinates of $T(u)$ with respect to the basis B."

Please Note

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For example if $u = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, the co-ordinate vector of u w.r.t. B is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

because

$$u = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$T(u)$ from page 13·18 is given by

$$T(u) = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

The co-ordinate vector of $T(u)$ w.r.t. β is

$$\begin{pmatrix} 6\frac{1}{2} \\ 1\frac{1}{2} \\ -3\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

because

$$T(u) = 6\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} -3\frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

The matrix A we are looking for takes the co-ordinate vector $\begin{pmatrix}1\\1\\0\end{pmatrix}$

and maps it to the vector $\begin{pmatrix}6\frac{1}{2}\\1\frac{1}{2}\\-3\frac{1}{2}\end{pmatrix}$ i.e.

$$A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}6\frac{1}{2}\\1\frac{1}{2}\\-3\frac{1}{2}\end{pmatrix}$$

Of course, the matrix A should map the corresponding co-ordinates for any vector in $\mathbb{R}^3$ and not just for $u=\begin{pmatrix}1\\2\\1\end{pmatrix}$. Our problem is to find A.

In order to find the matrix A, we proceed as follows:

Rather than starting with a u in $\mathbb{R}^3$ and finding the co-ordinates of u and $T(u)$ with respect to the basis B, let us now ask the inverse question.

"Which vector in $\mathbb{R}^3$ has the co-ordinates $\begin{pmatrix}1\\0\\0\end{pmatrix}$ with respect to basis B?"

Answer to this question is of course the vector

$$1\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\1\end{pmatrix} + 0\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Let us choose

$$u = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

and find $T(u)$. From page 13·18

$$T(u) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Let us now find the co-ordinates of $T(u)$ w.r.t. the basis B.

$$T(u) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Hence $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ is the co-ordinate vector of $T(u)$ w.r.t. the basis B.

The matrix 'A', maps the co-ordinate vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to the co-ordinate vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Hence the first column of the A matrix is $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

Remark: The second and third column of the A matrix can be computed likewise by starting with the co-ordinate vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Algorithm to calculate the A matrix can be summarized as follows:

① Let us denote the basis B as $B = \{b_1, b_2, b_3\}$.

② The i^{th} column of the A matrix is the co-ordinate vector of $T(b_i)$ w.r.t the basis B.

Summary of the algorithm.

Writing the matrix A as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we have

$$T(b_1) = B \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

This is because the 1st column of A matrix is the co-ordinate of $T(b_1)$ w.r.t the basis B.

Likewise

$$T(b_2) = B \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

$$T(b_3) = B \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Combining, we get

$$\begin{bmatrix} T(b_1) & T(b_2) & T(b_3) \end{bmatrix} = B A.$$

i.e.

$$\begin{array}{c} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 2 & 5 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ T(b_1) & T(b_2) & T(b_3) \end{array} \end{array} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} A$$

Hence

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 2 & 5 \end{pmatrix}$$

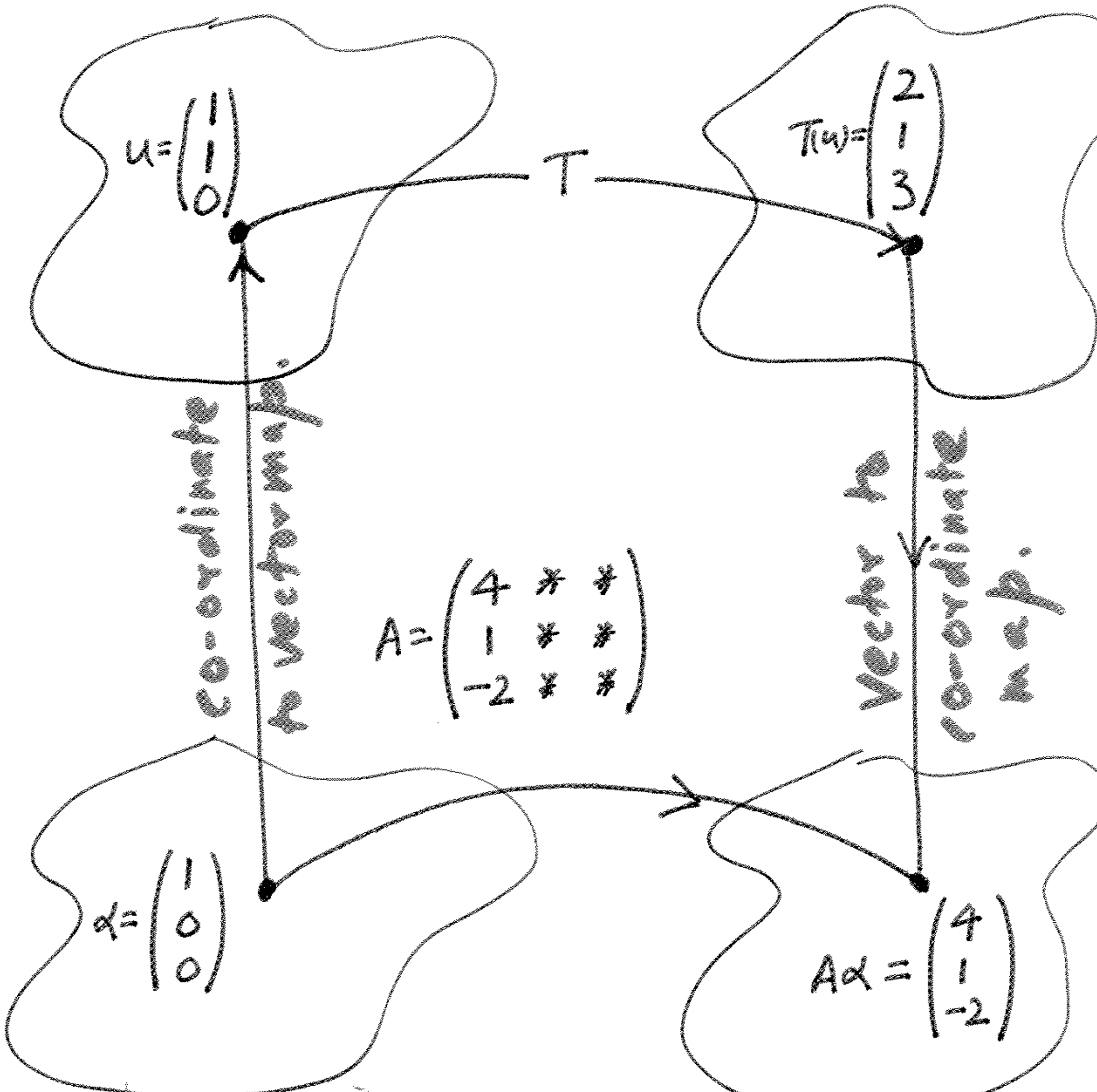
$$= \begin{pmatrix} 4 & 5/2 & 11/2 \\ 1 & 1/2 & 5/2 \\ -2 & -3/2 & -5/2 \end{pmatrix}$$

you should now be able to check that this A matrix satisfies the condition on page 13·23 ie A maps the co-ordinate vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} 6\frac{1}{2} \\ 1\frac{1}{2} \\ -3\frac{1}{2} \end{pmatrix}$.

The matrix A is called the "matrix representation" of the transformation T with respect to basis $\mathcal{B}$.

If we write $\mathcal{B} = (b_1\ b_2\ b_3)$ then

$$A = \mathcal{B}^{-1}\left[T(b_1)\ T(b_2)\ T(b_3)\right]$$



Steps to calculate the first column of the A matrix.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$$ ← Basis is same as in page 13·19.

Example 11 :—

In this example, we consider the same transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ as in example 10. But this time we consider a different basis, B_1 where

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Repeating the calculation in Example 10, we obtain a matrix representation A_1 of T with respect to the new basis B_1. The matrix A_1 turns out to be

$$A_1 = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & -3 \\ 1 & 3 & 3 \end{pmatrix}$$

see next page

As in page 13·30, if we write

$$B_1 = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$

then

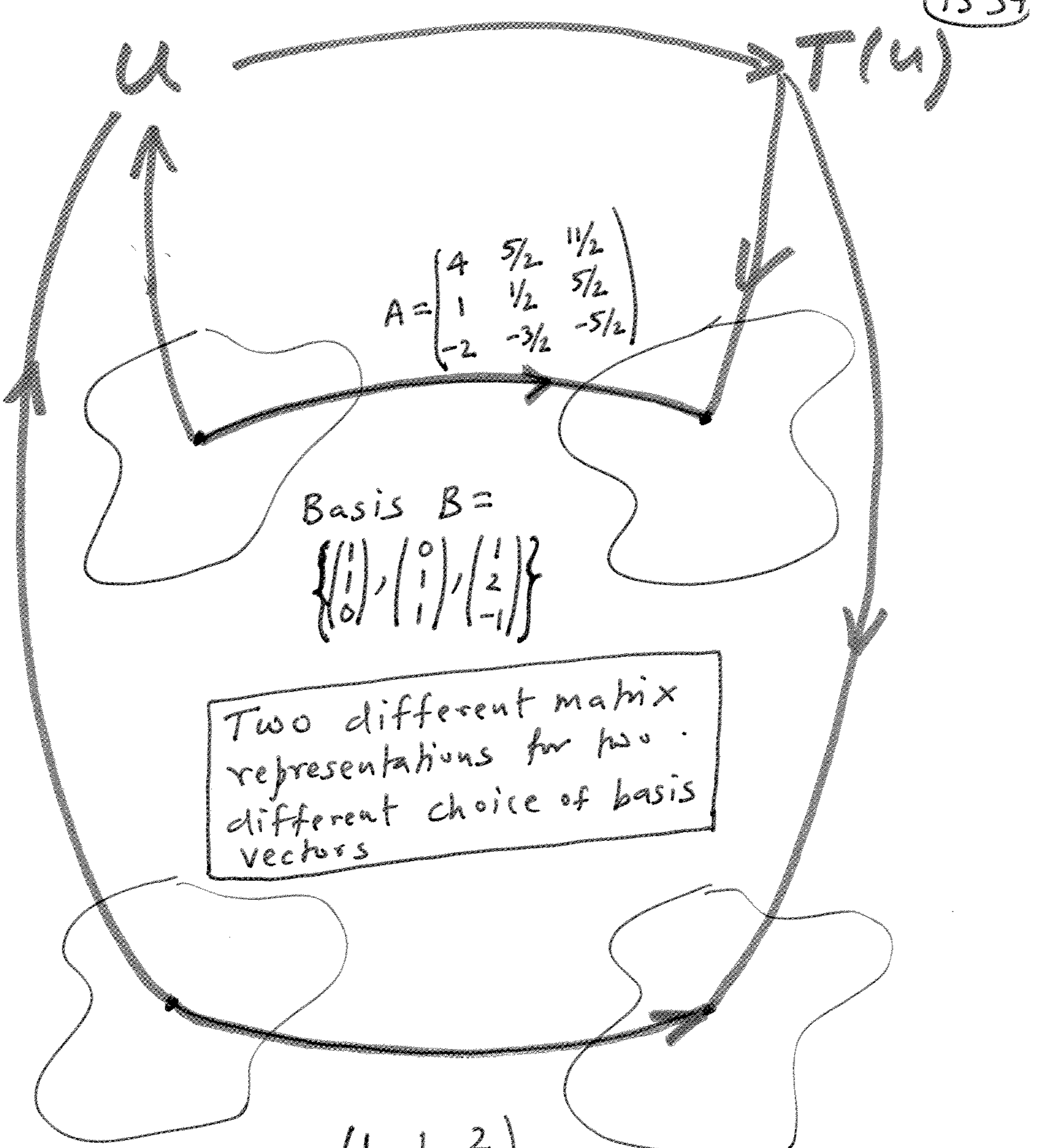
$$A_1 = B_1^{-1} \begin{pmatrix} T(c_1) & T(c_2) & T(c_3) \end{pmatrix}$$

Of course we have

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad c_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad c_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$T(c_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad T(c_2) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad T(c_3) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$\therefore A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & -3 \\ 1 & 3 & 3 \end{pmatrix}$$



$$A = \begin{pmatrix} 4 & 5/2 & 11/2 \\ 1 & 1/2 & 5/2 \\ -2 & -3/2 & -5/2 \end{pmatrix}$$

Basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$

Two different matrix representations for two different choice of basis vectors

$$A_1 = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & -3 \\ 1 & 3 & 3 \end{pmatrix}$$

Basis $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

We have thus shown two different matrix representations A & A_1 for the same linear transformation T with respect to two different bases.

It can be shown that the matrices A & A_1 are similar i.e $\exists$ a matrix P :

$$PAP^{-1} = A_1 .$$

The matrix P is given by

$$P = B_1^{-1} B$$

where

$$B_1 = (c_1 \ \ c_2 \ \ c_3)$$

$$B = (b_1 \ \ b_2 \ \ b_3)$$

comes from the basis vectors.

$$B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

Hence

$$P = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

Summary:

All matrix representations of a linear transformation are similar matrices. We illustrated this by choosing 2 representations.